Existence of positive periodic solutions of functional difference equations with sign-changing terms

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ABSTRACT. This paper is concerned with the nonlinear functional difference equation
\[ \Delta x(n) = -a(n)x(n) + \lambda h(n)f(n, x(n - \tau(n))), \]
where \( h \) and \( f \) may change sign. Sufficient conditions for the existence of at least one positive \( T \)-periodic solution are established.

1. INTRODUCTION

In the past ten years, there has been a large amount of papers concerned with the higher order difference equation
\[ x(n + 1) = f(n, x(n), \ldots, x(n - k)), \quad n \in \mathbb{Z}. \]
See [1-11] and the references therein. The most discussed problem is as follows:

- Under what conditions equation (1.1) has positive \( T \)-periodic solutions or \( T \)-periodic solutions?

For example, in [5], [6], [8-11], the authors studied the existence of positive periodic solutions of the special cases of equation (1.1) of the forms
\[ x(n + 1) = b(n)x(n) + \lambda h(n)f(x(n - \tau(n))), \]
and
\[ x(n + 1) = b(n)x(n) - \lambda h(n)f(x(n - \tau(n))). \]

In all above mentioned papers, it is supposed that \( h(n) \) and \( b(n) \) are nonnegative \( T \)-periodic sequences with \( 1 > b(n) > 0, f(x) \) is a nonnegative function.

In applications, \( h, b \) and \( f \) may change sign [4]. The discrete model of hematopoiesis with negative feedback [10] is as follows:
\[ \Delta x(n) = -a(n)x(n) + p(n) \frac{1}{1 + x^m(n - kT)} , \quad n \in \mathbb{Z}, \]
and the discrete delay survival red blood cells model [6, 11] is
\[ \Delta x(n) = -a(n)x(n) + p(n)e^{-r(n)x(n-kT)} , \quad n \in \mathbb{Z}. \]

In models (1.2) and (1.3), if cells are transported into the circulation, then \( a(n) < 0 \), furthermore the flux \( p(n) \) may be nonpositive. Hence it is interesting to consider the existence of positive periodic solutions of (1.2) and (1.3) when \( a(n) \) or \( p(n) \) changes sign.

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In this paper, we investigate nonlinear functional difference equation of the form

$$\Delta x(n) = -a(n)x(n) + \lambda h(n)f(n, x(n - \tau(n))), \quad n \in \mathbb{Z},$$  \hspace{1cm} (1.4)

where \( \lambda > 0 \) is a parameter, \( a(n), h(n) \) and \( \tau(n) \) are \( T \)-periodic sequences, \( f(n, x) \) is continuous in \( x \) and \( T \)-periodic in \( n \).

The purpose of this paper is to establish existence results for positive \( T \)-periodic solutions of equation (1.4) when \( h(n) \) and or \( f(n, x) \) changes sign. Applying the main results, the existence of positive periodic solutions of (1.2) and (1.3) are established when \( a(n), p(n) \) change sign.

The remainder of this paper is divided into two sections. In Section 2, the main results are given and proved. Some applications to illustrate the main results are shown in Section 3.

## 2. Main Results

The following assumptions should be used in the main results:

\((A_1)\) \( f : \mathbb{Z} \times [0, +\infty) \to \mathbb{R} \) is continuous and there exists a constant \( M > 0 \) such that \( f(n, x) \geq (\not\equiv) - M \) for \( (n, x) \in \mathbb{Z} \times [0, +\infty) \).

\((A_2)\) \( f : \mathbb{Z} \times [0, +\infty) \to (0, +\infty) \) is continuous and \( \min_{n \in [0, T - 1]} f(n, 0) > 0 \) and denote \( \alpha = \frac{\max_{n \in [0, T - 1]} f(n, 0)}{\min_{n \in [0, T - 1]} f(n, 0)} \).

\((A_3)\) \( h : \mathbb{Z} \to [0, +\infty) \) satisfies \( \sum_{n=0}^{T-1} h(n) > 0 \);

\((A_4)\) \( h : \mathbb{Z} \to \mathbb{R} \) satisfies that there exists a constant is \( k > 2\alpha - 1, \alpha \) is defined in \((A_2)\), such that

$$\sum_{s=n}^{n+T-1} G(n, s)h^+(s)ds \geq k \sum_{s=n}^{n+T-1} G(n, s)h^-(s)ds \text{ for all } n \in [0, T - 1],$$  \hspace{1cm} (2.5)

where \( \alpha \) is defined in \((A_2)\), \( h^+(n) = \max\{0, h(n)\} \) and \( h^-(n) = \max\{0, -h(n)\} \) and

$$G(n, k) = \left[ \prod_{s=k+1}^{n+T-1} (1 - a(s)) \right] \left[ 1 - \prod_{j=0}^{T-1} (1 - a(s)) \right]^{-1}.$$

\((A_5)\) \( \lim_{n \to +\infty, n \in [0, T - 1]} \frac{f(n, x)}{x} = N \in (0, +\infty) \).

\((A_6)\) \( a(n) \) satisfies \( a(n) < 1 \) and \( \prod_{s=n}^{n+T-1} (1 - a(s)) < 1 \).

Denote \([0, T - 1] = \{0, 1, \ldots, T - 1\}\). Let \( X \) be the set of all real \( T \)-periodic sequences \( \{x(n)\}_{n=-\infty}^{+\infty} \) and endowed with the norm \( ||x|| = \max_{n \in [0, T - 1]} |x(n)| \). Then \( X \) is a Banach space. Let

\[ K = \{ x \in X : x(n) \geq \sigma ||x||, \ n \in [0, T - 1] \}, \]
where \(\sigma\) is defined by \(\sigma = \begin{cases} 1, & T = 1, \\ \prod_{j=0}^{T-1} a^-(j)[a^+(s)]^{-1}, & T \geq 2, \end{cases}\)
and \(a^+(n) = \max\{1, 1 - a(n)\}\), \(a^-(n) = \min\{1, 1 - a(n)\}\). Then \(K\) is a cone of space \(X\). Suppose that \((A_6)\) holds. It is easy to prove that \(x \in X\) is a \(T\)-periodic solution of equation (1.4) if and only if

\[
x(n) = \sum_{s=n}^{n+T-2} G(n, s)\mu(s) + \frac{1}{1 - \prod_{j=0}^{T-1} (1 - a(s))}\mu(n-1), \quad n \in \mathbb{Z},
\]

where \(G(n, k)\) is defined in (2.6). Furthermore, we have

\[
G(n, s) \geq \frac{\prod_{j=0}^{T-1} a^-(j)}{1 - \prod_{j=0}^{T-1} (1 - a(j))}, \quad G(n, s) \leq \frac{\prod_{j=0}^{T-1} a^+(j)}{1 - \prod_{j=0}^{T-1} (1 - a(j))}.
\]

**Theorem 2.1.** Suppose that \((A_1), (A_3), (A_5)\) and \((A_6)\) hold. Then equation (1.4) has at least one positive solution if \(\lambda \in (A, B)\), \(A\) and \(B\) are defined by

\[
A = \frac{2}{\sigma N} \frac{1 - \prod_{j=0}^{T-1} (1 - a(j))}{\prod_{j=0}^{T-1} a^-(j) \sum_{n=0}^{T-1} h(n)},
\]

\[
B = \min \left\{ \frac{1}{M_1} \frac{1 - \prod_{j=0}^{T-1} (1 - a(j))}{\prod_{j=0}^{T-1} a^+(j) \sum_{n=0}^{T-1} h(n)}, \frac{\sigma}{2M} \frac{1 - \prod_{j=0}^{T-1} (1 - a(j))}{\prod_{j=0}^{T-1} a^+(j) \sum_{n=0}^{T-1} h(n)} \right\},
\]

where

\[
M_1 = \max_{(n,x) \in [0,T-1] \times [0,1]} g(n, x), \quad r = \frac{BM\|w\|}{\sigma},
\]

\[
w(n) = \sum_{s=n}^{n+T-2} G(n, s)h(s) + \frac{1}{1 - \prod_{j=0}^{T-1} (1 - a(s))} h(n-1),
\]

\[
g(n, x) = \begin{cases} f(n, x) + M, & (n, x) \in [0, T-1] \times [0, +\infty), \\ f(n, 0) + M, & (n, x) \in [0, T-1] \times (-\infty, 0). \end{cases}
\]

**Proof.** By the definition of \(w(n)\) and \(z(n) = \lambda Mw(n), (A_3), (A_6)\), we get that \(z(n) \geq 0\) for all \(n \in \mathbb{Z}\). It is easy to see

\[
(2.8) \quad \Delta z(n) = -a(n)z(n) + \lambda Mh(n), \quad n \in \mathbb{Z}.
\]

Consider the following equation

\[
(2.9) \quad \Delta y(n) = -a(n)y(n) + \lambda h(n) g \left( n, y(n - \tau(n)) - z(n - \tau(n)) \right), \quad n \in \mathbb{Z}.
\]
It is easy to see from (1.2), (2.8) and (2.9) that equation (1.4) has a positive $T$-periodic solution $x(n)$ if and only if $x(n) + z(n) := y(n)$ is a $T$-periodic solution of the equation (2.9) and $y(n) > z(n)$ for all $n \in \mathbb{Z}$.

Define the operator $Q$ by

$$(Qy)(n) = \lambda \sum_{s=n}^{n+T-1} G(n, s) h(s) g(s, y(s - \tau(s)) - z(s - \tau(s)))$$

for $y \in X$. It follows from $(A_1), (A_3), (A_6)$ that $Q$ is completely continuous, $QK \subset K$ and $y$ is a solution of (2.9) if and only if $y$ is a solution of the operator equation $Qy = y$. Let $\lambda$ be fixed and $A < \lambda \leq B$. We do two steps.

**Step 1.** Let $\Omega_1 = \{ y \in X : ||y|| < r \}$. For $y \in K \cap \partial \Omega_1$, one has $||y|| = r$. Then $y(n - \tau(n)) - z(n - \tau(n)) \leq y(n - \tau(n)) \leq ||y|| = r$, and $y(n - \tau(n)) - z(n - \tau(n)) \geq \sigma ||y|| - ||z|| \geq \sigma r - \lambda M \|w\| \geq \sigma r - BM \|w\| = 0$. Using $\lambda \leq B$, one gets that

$$(Qy)(n) \leq \lambda M (\prod_{j=0}^{T-1} a^+(j)) \sum_{s=0}^{T-1} h(s) \leq r = ||y||,$$

i.e., $||Qy|| \leq ||y||$ for all $y \in K \cap \partial \Omega_1$.

**Step 2.** We consider two cases: $N \in (0, +\infty)$ and $N = +\infty$. Suppose $N \in (0, +\infty)$. Since $\lambda > A$, we choose $\varepsilon > 0$ such that

$$\frac{\lambda(N - \varepsilon)\sigma}{2} > \frac{\prod_{j=0}^{T-1} a^-(j)}{1 - \prod_{j=0}^{T-1} (1 - a(j))} \sum_{s=0}^{T-1} h(s) \geq 1. \tag{2.10}$$

Since $\lambda \leq B$, we get

$$\lambda \leq \frac{\sigma}{2M} \frac{\prod_{j=0}^{T-1} (1 - a(j))}{\prod_{j=0}^{T-1} a^+(j)} \frac{1}{\sum_{n=0}^{T-1} h(n)}. \tag{2.11}$$

It is easy to see that

$$\lim_{x \to +\infty} \min_{n \in [0, T-1]} \frac{g(n, x)}{x} = \lim_{x \to +\infty} \min_{n \in [0, T-1]} \frac{f(n, x) + M}{x} = N.$$

For above $\varepsilon > 0$, use $(A_5)$, choose $\bar{R} > \max \{1, r\}$ sufficiently large such that

$$\frac{g(n, x)}{x} = \frac{f(n, x) + M}{x} \geq N - \varepsilon \quad \text{for} \quad (n, x) \in [0, T - 1] \times \left[\frac{\sigma \bar{R}}{2}, +\infty\right).$$

Set $\Omega_2 = \{ y \in X : ||y|| < \bar{R} \}$. Using (2.11) and $\bar{R} > 1$, we find, for $y \in K \cap \partial \Omega_2$, that

$$y(n) - z(n) \geq \sigma \bar{R} - \frac{\sigma}{2} \geq \frac{\sigma \bar{R}}{2}.$$
Hence

\[
(Qy)(n) \geq \frac{\lambda \sigma \mathcal{R}(N - \varepsilon)}{2} \frac{\prod_{j=0}^{T-1} a^{-}(j)}{1 - \prod_{j=0}^{T-1} (1 - a(j))} \sum_{s=0}^{T-1} h(s) \geq \mathcal{R} = \|y\|,
\]

i.e., \(\|Qy\| \geq \|y\|\) for \(y \in K \cap \partial \Omega_2\).

Suppose \(N = +\infty\), choose \(M' > \max\{1, r\}\) such that

\[
\text{(2.12)} \quad \frac{\lambda M' \sigma}{2} \frac{\prod_{j=0}^{T-1} a^{-}(j)}{1 - \prod_{j=0}^{T-1} (1 - a(j))} \sum_{s=0}^{T-1} h(s) \geq 1.
\]

Choose \(\mathcal{R} > \max\{1, r\}\) sufficiently large such that

\[
g(n, x) = f(n, x) + M' f_{or} (n, x) \in [0, T - 1] \times \left[\frac{\sigma \mathcal{R}}{2}, +\infty\right).
\]

Using (2.11) and (2.12), let \(\Omega_2 = \{ y \in X : \|y\| < \mathcal{R} \}\). By similar discussion above, we get \(\|Qy\| \geq \|y\|\) for \(y \in K \cap \partial \Omega_2\).

Hence Krasnoselskii fixed point theorem [2] implies that \(Q\) has at least one fixed point \(y\) such that \(r \leq \|y\| \leq \mathcal{R}\). Now, we prove that \(y - z\) is positive.

In fact, one sees that

\[
y(n) - z(n) = y(n) - \lambda Mw(n) > \sigma \|y\| - \lambda M \|w\| \geq \sigma r - BM \|w\| = 0.
\]

Then \(y(n) > \lambda Mw(n) = z(n)\) for all \(n \in Z\). So \(x(n) = y(n) - z(n)\) is a positive \(T\)-periodic solution of equation (1.4). The proof is complete. \(\square\)

**Theorem 2.2.** Suppose that \((A_2), (A_4)\) and \((A_6)\) hold. Then there is a positive number \(\lambda^*\) such that equation (1.4) has at least one positive \(T\)-periodic solution for \(\lambda \in (0, \lambda^*)\).

**Proof.** We divide the proof into two steps.

**Step 1.** Prove that for every \(0 < \delta < 1\), there exists a positive number \(\lambda\) such that, for \(\lambda \in (0, \lambda]\), the integral equation

\[
(2.13) \quad x(n) = \lambda \sum_{s=n}^{n+T-1} G(n, s)h^+(s)f(s, x(s - \tau(s))) := (Hx)(n), \quad n \in Z
\]

has a positive \(T\)-periodic solution \(\pi_{\lambda}\) such that

\[
(2.14) \quad \|\pi_{\lambda}\| \to 0, \quad \text{as} \quad \lambda \to 0, \quad \pi_{\lambda}(n) \geq \lambda \delta \inf_{n \in [0, T-1]} f(n, 0) \|p\|,
\]

where \(p(n) = \sum_{s=n}^{n+T-1} G(n, s)h^+(n)\).

Let \(\overline{f}(x) = \max_{0 \leq s \leq x} \max_{n \in [0, T-1]} f(n, s)\). Since \(0 < \delta < 1\) and \((A_2)\) holds, we get

\[
f(n, 0) > \delta \inf_{n \in [0, T-1]} f(n, 0).
\]
Since \( f(n, x) \) is continuous, we can choose \( \varepsilon > 0 \) such that
\[
(2.15) \quad f(n, x) \geq \delta \inf_{n \in [0, T-1]} f(n, 0) \text{ for } 0 \leq x \leq \varepsilon.
\]

Let \( \bar{\lambda} = \frac{\varepsilon}{2\|p\|\overline{f}(\varepsilon)} \). Suppose that \( 0 < \lambda < \bar{\lambda} \). It follows from the definition of \( \overline{f}(x) \) and
\[
\min_{n \in [0, T-1]} f(n, 0) > 0 \text{ and (2.15) that}
\]
\[
\lim_{x \to 0^+} \frac{\overline{f}(x)}{x} = \lim_{x \to 0^+} \frac{\max_{0 \leq s \leq x} \max_{n \in [0, T-1]} f(n, s)}{x} = +\infty.
\]
This together with \( \overline{f}(\varepsilon) \) imply that there is \( r_\lambda \in (0, \varepsilon) \) such that
\[
(2.16) \quad \frac{\overline{f}(r_\lambda)}{r_\lambda} = \frac{1}{2\lambda\|p\|}.
\]

Then \( r_\lambda = 2\lambda\|p\|\overline{f}(r_\lambda) \). It follows that \( r_\lambda \to 0 \) as \( \lambda \to 0 \).

Now, consider the homotopy equation \( u = \theta Hu, \theta \in [0, 1] \). Let \( u \in X \) and \( \theta \in [0, 1] \) be such that \( u = \theta Hu \). We claim that \( \|u\| \neq r_\lambda \). Set
\[
w(n) = \theta \lambda \sum_{s=n}^{n+T-1} G(n, s)h^+(s)\overline{f}(\|u\|) = \theta \lambda \overline{f}(\|u\|)p(n).
\]

It follows from \( \overline{f}(x) = \max_{0 \leq s \leq x} \max_{n \in [0, T]} f(n, s) \) that \( f(n, u(n - \tau(n))) \leq \overline{f}(\|u\|) \). Thus \( u(n) \leq w(n) \) for all \( n \in [0, T - 1] \). Moreover, we have \( \|u\| \leq \lambda\|p\|\overline{f}(\|u\|) \). Then
\[
(2.17) \quad \frac{\overline{f}(\|u\|)}{\|u\|} \geq \frac{1}{\lambda\|p\|}.
\]

It follows from (2.16) and (2.17) that \( \|u\| \neq r_\lambda \). Thus by Leray-Schauder fixed point Theorem [2], \( H \) has a fixed point \( \varpi_\lambda \) with \( \|\varpi_\lambda\| \leq r_\lambda < \varepsilon \). Together with (15), we get
\[
(2.18) \quad \varpi_\lambda(n) \geq \lambda \delta \min_{n \in [0, T-1]} f(n, 0)p(n), \quad n \in Z.
\]

**Step 2.** Prove that the equation (1.4) has a positive \( T \)-periodic solution.

Let
\[
(2.19) \quad q(n) = \sum_{s=n}^{n+T-1} G(n, s)h^-(s).
\]

Then \( q(n) \geq 0 \). \((A_4)\) implies \( p(n)/q(n) \geq k \).

It follows from the definition of \( \alpha \) that \( \alpha \geq 1 \). Since \( k > 2\alpha - 1 \), we can choose \( d \in (0, 1) \) such that \( kd > 1 \) and \( (2\alpha - 1)d < 1 \). Since \( f(n, 0) < kd \max_{n \in [0, T-1]} f(n, 0) \)
and \( f(n, x) \) is continuous, there is \( c > 0 \) such that
\[
f(n, y) \leq kd \max_{n \in [0, T-1]} f(n, 0), \quad y \in [0, c].
\]
Together with (A4), one gets

\[ q(n)f(n, y) \leq d\alpha\|p\| \min_{n \in [0, T-1]} f(n, 0), \quad n \in \mathbb{Z}, \quad y \in [0, c]. \]

Fix \( \delta \in (2(\alpha - 1)d, 1) \), it follows from Step 1 that there is \( \lambda^* > 0 \) such that (2.13) has a positive \( T \)-periodic solution \( x_\lambda \) for \( \lambda \in (0, \lambda^*) \) with \( \|x_\lambda\| \to 0 \) as \( \lambda \to 0 \).

Without loss of generality, suppose \( \lambda^* > 0 \) sufficiently small such that

\[ (2.20) \quad \|x_\lambda\| + \lambda\delta \min_{n \in [0, T-1]} f(n, 0)\|p\| \leq c, \quad \lambda \in (0, \lambda^*), \]

and

\[ (2.21) \quad |f(n, x) - f(n, y)| \leq \min_{n \in [0, T-1]} f(n, 0)\frac{\delta - d}{2}, \quad n \in [0, T - 1] \]

for \( x, y \in [-c, c] \) with \( |x - y| \leq \lambda^*\delta \min_{n \in [0, T-1]} f(n, 0)\|p\| \).\n
Let \( \lambda \in (0, \lambda^*) \), we look for a solution \( y_\lambda \) of the following integral equation

\[ y(n) = \lambda \sum_{s=n}^{n+T-1} G(n, s) \times \left\{ h^+(s) \left[ f \left( s, x_\lambda(s - \tau(s)) + y(s - \tau(s)) \right) - f \left( s, x_\lambda(s - \tau(s)) \right) \right] \right\} \]

\[ -h^-(s)f \left( s, x_\lambda(s - \tau(s)) + y(s - \tau(s)) \right) \right\} =: H'y(n). \]

Since \( f \) is continuous, \( H' \) is completely continuous. Let \( y \in \mathcal{X} \) and \( \theta \in [0, 1] \) be such that \( y = \theta H'y \). We claim that \( \|y\| \neq \lambda\delta \min_{n \in [0, T-1]} f(n, 0)\|p\| \). Suppose to the contrary that \( \|y\| = \lambda\delta \min_{n \in [0, T-1]} f(n, 0)\|p\| \). Then, by (2.20), we get

\[ (2.22) \quad \|x_\lambda + y\| \leq \|x_\lambda\| + \|y\| \leq \|x_\lambda\| + \lambda\delta \min_{n \in [0, T-1]} f(n, 0)\|p\| \leq c. \]

Now, we get that

\[ \|(x_\lambda + y) - x_\lambda\| = \|y\| = \lambda\delta \min_{n \in [0, T-1]} f(n, 0)\|p\| \leq \lambda^*\delta \min_{n \in [0, T-1]} f(n, 0)\|p\|. \]

It follows from (2.21) that

\[ (2.23) \quad |f(\lambda\delta + 2(\alpha - 1)d)\min_{n \in [0, T-1]} f(n, 0)\|p\| \leq \lambda^*\delta \min_{n \in [0, T-1]} f(n, 0)\|p\|. \]

Using \( y = \theta H'y \) and \( q(n)f(n, y) \leq d\alpha\min_{n \in [0, T-1]} f(n, 0)\|p\| \), we get

\[ \|y(n)\| \leq \lambda \delta + \frac{2(\alpha - 1)d}{2} \min_{n \in [0, T-1]} f(n, 0)\|p\|. \]

In particular, one has

\[ \|y\| \leq \lambda \delta + \frac{2(\alpha - 1)d}{2} \min_{n \in [0, T-1]} f(n, 0)\|p\|, \]

which contradicts \( \|y\| = \lambda\delta \min_{n \in [0, T-1]} f(n, 0)\|p\| \) since \( (2\alpha - 1)d < \delta \), and the claim is proved. Thus, by Leray-Schauder fixed point Theorem [2], \( H' \) has a fixed point.
It follows from (2.18) and (2.24) that

\[ x_\lambda(n) + y_\lambda(n) \geq \lambda \delta - (2\alpha - 1)d_{\min} \min_{n \in [0, T-1]} f(n, 0)p(n) > 0, \]

It is easy to get that \( x_\lambda + y_\lambda \) is a positive \( T \)-periodic solution of equation (1.4). The proof is complete. \( \square \)

3. APPLICATIONS

Consider the discrete model of hematopoiesis with negative feedback and the discrete delay survival red blood cells model

\[ \Delta x(n) = -a(n)x(n) + \lambda \frac{h(n)}{1 + x^m(n - \tau(n))}, \quad n \in \mathbb{Z}, \]

\[ \Delta x(n) = -a(n)x(n) + \lambda h(n)e^{-r(n)}x(n - \tau(n)), \quad n \in \mathbb{Z}, \]

where \( \lambda > 0, m > 0 \) is an integer, \( a(n), h(n), \tau(n), r(n) \) are \( T \)-periodic sequences with period \( T \geq 1 \).

(\( A_7 \)) \( h : \mathbb{Z} \to \mathbb{R} \) satisfies that there exists a constant is \( k > 1 \) such that

\[ \sum_{s=n}^{n+T-1} G(n, s)h^+(s)ds \geq k \sum_{s=n}^{n+T-1} G(n, s)h^-(s)ds \text{ for all } n \in [0, T-1], \]

where \( h^+(n), h^-(n) \) and \( G(n, k) \) are defined in (2.6). Using Theorem 2.2, we get the following Corollaries.

**Corollary 3.1.** Suppose that (\( A_7 \)) and (\( A_6 \)) hold. Then there is a positive number \( \lambda^* \) such that equation (3.25) has at least one positive \( T \)-periodic solution for \( \lambda \in (0, \lambda^*) \).

**Corollary 3.2.** Suppose that (\( A_7 \)) and (\( A_6 \)) hold. Then there is a positive number \( \lambda^* \) such that equation (3.26) has at least one positive \( T \)-periodic solution for \( \lambda \in (0, \lambda^*) \).

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