A quicker convergence toward the $\gamma$ constant with the logarithm term involving the constant $e$

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ABSTRACT. We introduce a new class of sequences of the form

$$\mu_n = \sum_{k=1}^{n} \frac{1}{k} + \ln(e^{a/(n+b)} - 1) - \ln a$$

which converge to the Euler-Mascheroni constant $\gamma$. Being preoccupied to accelerate the classical sequence convergent toward $\gamma$, Batir [J. Ineq. Pure Appl. Math. 6 (2005) no. 4 Art 103] and Alzer [Expo. Math. 24 (2006) 385-388] studied the case $a = b = 1$ and we show in this paper that the fastest sequence $(\mu_n)_{n \geq 1}$ is obtained for $a = 1/\sqrt{2}, b = (2 + \sqrt{2})/4$. For these values, accurate approximations of $\gamma$ can be constructed, as numerical computations made in the final part of this paper show. We also solve an open problem about the rate of convergence of some sequences defined by Batir.

1. INTRODUCTION

One of the most useful constant in mathematics is the limit of the sequence

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n,$$

denoted $\gamma = 0.57721566490153286...$. It is called the Euler-Mascheroni constant after the Swiss mathematician Leonhard Euler (1707-1783) and the Italian mathematician Lorenzo Mascheroni (1750-1800).

The sequence $(\gamma_n)_{n \geq 1}$ and the constant $\gamma$ have numerous applications in many areas of mathematics and science in general, as analysis, theory of probability, physics, applied statistics, special functions, or number theory.

The starting point of this paper is the works of Batir [4] and Alzer [2], where estimates of the form

$$\alpha \leq \sum_{k=1}^{n} \frac{1}{k} + \ln \left(e^{1/(n+1)} - 1\right) < \beta, \ n \geq 1$$

were established. More precisely, Batir [4] proved (1.1) with the bounds $\alpha = \ln(\pi^2/6) = 0.4977...$, $\beta = \gamma$, while Alzer [2] found and showed that the best possible constants are $\alpha = 1 + \ln(\sqrt{e} - 1) = 0.5672...$ and $\beta = \gamma = 0.577...$.

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Motivated by these results, we introduce in this paper the following class of sequences of the form

$$\mu_n(a, b) = \sum_{k=1}^{n} \frac{1}{k} + \ln \left( e^{a/(n+b)} - 1 \right) - \ln a,$$

where $a, b$ are real parameters, $a > 0$.

In the last part of [4], the author offered numerical experiments which show that the sequence $\mu_n(1, 1)$ gives good results in the problem of estimation of $\gamma$, but theoretical results about the speed of convergence are not given.

Numerical computations from [4] prove the superiority of the approximation $\gamma \approx \mu_n(1, 1)$ over the classical approximation $\gamma \approx \gamma_n$ and that it is slightly more accurate than the approximation $\gamma \approx R_n$, where

$$R_n = \sum_{k=1}^{n} \frac{1}{k} - \ln \left( n + \frac{1}{2} \right)$$

is DeTemple’s sequence (see [6], [7]).

These facts become natural when we prove later that $(\mu_n(1, 1))_{n \geq 1}$ converges to $\gamma$ like $n^{-2}$, as the sequence $(R_n)_{n \geq 1}$ does, since

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2} \quad (\text{e.g., [6]}).$$

For the classical sequence $(\gamma_n)_{n \geq 1}$, we have, for example,

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n} \quad \text{(R. M. Young [27])},$$

or other estimates given in [1], [3], [5], [9], [10], [11], [12], [13], [25], [26] which show that $(\gamma_n)_{n \geq 1}$ converges to $\gamma$ like $n^{-1}$.

We prove that among the sequences $(\mu_n(a, b))_{n \geq 1}$, the privileged one

$$\mu_n \left( \frac{\sqrt{2}}{2}, \frac{2 + \sqrt{2}}{4} \right)$$

offers the best approximations of $\gamma$, since

$$\lim_{n \to \infty} n^3 \left( \mu_n \left( \frac{\sqrt{2}}{2}, \frac{2 + \sqrt{2}}{4} \right) - \gamma \right) = \frac{\sqrt{2}}{96}.$$

Our study is based on the following result, which gives a method for measuring the speed of convergence:

**Lemma 1.1.** If $(\omega_n)_{n \geq 1}$ is convergent to zero and there exists the limit

$$\lim_{n \to \infty} n^k(\omega_n - \omega_{n+1}) = l \in \mathbb{R},$$

with $k > 1$, then there exists the limit:

$$\lim_{n \to \infty} n^{k-1}\omega_n = \frac{l}{k-1}.$$
This lemma was used by Mortici [8]-[24] for constructing asymptotic expansions or to accelerate some convergences. For proof, see, e.g., [14]-[15].

We can see from Lemma 1.1 that the speed of convergence of the sequence \((\omega_n)_{n \geq 1}\) is as higher as the value \(k\) satisfying (1.3) is greater.

2. The results

For every real parameters \(a, b,\) with \(a > 0,\) we define the sequence

\[
\mu_n = \sum_{k=1}^{n} \frac{1}{k} + \ln \left( e^{a/(n+b)} - 1 \right) - \ln a.
\]

As we are interested to compute a limit of the form (1.3), it is advantageous for us to write the difference \(\mu_n - \mu_{n+1}\) as power series of \(n^{-1}\). This can be easily done using a computer software, thus

\[
(2.4) \quad \mu_n - \mu_{n+1} = \left( \frac{1}{2} a - b + \frac{1}{2} \right) \frac{1}{n^2} + \left( \frac{1}{12} a^2 - ab - \frac{1}{2} a + b^2 + b - \frac{2}{3} \right) \frac{1}{n^3} + \left( -\frac{1}{4} a^2 b - \frac{1}{8} a^2 + \frac{3}{2} a b^2 + \frac{3}{2} a b - \frac{1}{2} a - b^3 - \frac{3}{2} b^2 - b + \frac{3}{4} \right) \frac{1}{n^4} + O \left( \frac{1}{n^5} \right).
\]

According to Lemma 1.1, we can see that the fastest sequence \((\mu_n)_{n \geq 1}\) is obtained in case when as many of the first coefficients of (2.4) are cancelled. As we have two parameters \(a, b,\) they produce the best result if and only if

\[
\begin{align*}
\frac{1}{2} a - b + \frac{1}{2} = 0, \\
\frac{1}{12} a^2 - ab - \frac{1}{2} a + b^2 + b - \frac{2}{3} = 0,
\end{align*}
\]

with the convenient solution \(a = \frac{\sqrt{2}}{2}, b = \frac{2 + \sqrt{2}}{4}.

Now, using Lemma 1.1 and relation (2.4), we can state the following

**Theorem 2.1.** i) If \(\frac{1}{2} a - b + \frac{1}{2} \neq 0,\) then the speed of convergence of the sequence \((\mu_n)_{n \geq 1}\) is \(n^{-1}\), since

\[
\lim_{n \to \infty} n^2 (\mu_n - \mu_{n+1}) = \frac{1}{2} a - b + \frac{1}{2} \text{ and } \lim_{n \to \infty} n (\mu_n - \gamma) = \frac{1}{2} a - b + \frac{1}{2} \neq 0.
\]

ii) If \(\frac{1}{2} a - b + \frac{1}{2} = 0,\) and \(\frac{1}{12} a^2 - ab - \frac{1}{2} a + b^2 + b - \frac{2}{3} \neq 0,\) then the speed of convergence of the sequence \((\mu_n)_{n \geq 1}\) is \(n^{-2}\), since

\[
\lim_{n \to \infty} n^3 (\mu_n - \mu_{n+1}) = \frac{1}{12} a^2 - ab - \frac{1}{2} a + b^2 + b - \frac{2}{3}
\]

and consequently,

\[
\lim_{n \to \infty} n^2 (\mu_n - \gamma) = \frac{1}{2} \left( \frac{1}{12} a^2 - ab - \frac{1}{2} a + b^2 + b - \frac{2}{3} \right) \neq 0.
\]
iii) If \( \frac{1}{2}a - b + \frac{1}{2} = 0 \), and \( \frac{1}{12}a^2 - ab - \frac{1}{2}a + b^2 + b - \frac{2}{3} = 0 \) (equivalent with \((a, b) = \left( \frac{\sqrt{2}}{2}, \frac{2 + \sqrt{2}}{4} \right)\)), then the speed of convergence of the sequence \((\mu_n)_{n \geq 1}\) is \(n^{-3}\), since
\[
\lim_{n \to \infty} n^4 (\mu_n - \mu_{n+1}) = \frac{\sqrt{2}}{32} \quad \text{and} \quad \lim_{n \to \infty} n^3 (\mu_n - \mu_{n+1}) = \frac{\sqrt{2}}{96}.
\]
For \((a, b) = \left( \frac{\sqrt{2}}{2}, \frac{2 + \sqrt{2}}{4} \right)\), relation (2.4) becomes
\[
\mu_n - \mu_{n+1} = \frac{\sqrt{2}}{32n^4} + O \left( \frac{1}{n^5} \right),
\]
so the assertion iii) of Theorem 2.1 is proved.

As the author of [4] declares, he had not investigated the rate of convergence of the sequence \(\mu_n(1, 1)\) and of the sequence
\[
\Gamma_n = \sum_{k=1}^{n} \frac{1}{k} + \frac{1}{2} \ln \left( \frac{e^{1/(n+1)} - 1}{n + 1/2} \right).
\]
We solve these problems using our powerful Lemma 1.1.

First, it is to be noted that for the sequence \(\mu_n(1, 1)\) we are in case ii) of Theorem 2.1, so
\[
\lim_{n \to \infty} n^2 (\mu_n(1, 1) - \gamma) = -\frac{1}{24},
\]
which explains the fact that numerical computations of [4] give comparative results for \(\mu_n(1, 1)\) and \(R_n\).

In order to find the speed of convergence of the sequence \((\Gamma_n)_{n \geq 1}\), we make appeal again to a computer software to get
\[
\Gamma_n - \Gamma_{n+1} = \frac{1}{16n^3} + O \left( \frac{1}{n^4} \right).
\]
From Lemma 1.1, it results that
\[
(2.5) \quad \lim_{n \to \infty} n^3 (\Gamma_n - \gamma) = \frac{1}{48}.
\]

3. NUMERICAL COMPUTATIONS

In [4], a numerical study proves that the approximation \(\gamma \approx \Gamma_n\) is more accurate than \(\gamma \approx \mu_n(1, 1)\) and \(\gamma \approx R_n\). It is expected to be so, now when we proved that \((\mu_n(1, 1))_{n \geq 1}\) and \((R_n)_{n \geq 1}\) converge to \(\gamma\) like \(n^{-2}\), while \((\Gamma_n)_{n \geq 1}\) has superior rate of convergence, as our sequence \(\left(\mu_n \left( \frac{\sqrt{2}}{2}, \frac{2 + \sqrt{2}}{4} \right) \right)_{n \geq 1}\) does.

Finally, comparing the rates of convergence (1.2) and (2.5), we deduce that our approximation \(\gamma \approx \mu_n \left( \frac{\sqrt{2}}{2}, \frac{2 + \sqrt{2}}{4} \right)\) is slightly more accurate than \(\gamma \approx \Gamma_n\), as
we can also see from the following table:

<table>
<thead>
<tr>
<th>n</th>
<th>$\Gamma_n - \gamma$</th>
<th>$\mu_n \left( \frac{\sqrt{2}}{2}, \frac{2+\sqrt{2}}{4} \right) - \gamma$</th>
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<td>$1.676 \times 10^{-5}$</td>
<td>$1.1807 \times 10^{-5}$</td>
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<td>$8.6183 \times 10^{-7}$</td>
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<td>$1.4402 \times 10^{-8}$</td>
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<td>$9.3431 \times 10^{-10}$</td>
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<td>1000</td>
<td>$2.0787 \times 10^{-11}$</td>
<td>$1.4698 \times 10^{-11}$</td>
</tr>
</tbody>
</table>

References

[9] Mortici, C. and Vernescu, A., An improvement of the convergence speed of the sequence $(\gamma_n)_{n \geq 1}$ converging to Euler’s constant, An. Şti. Univ. Ovidius Constanța 13 (2005), no. 1, 97-100
A quicker convergence


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