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A unified treatment of the modified Newton and chord methods

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ABSTRACT.
The aim of this paper is to obtain a unified treatment of some iterative methods. We obtain some conditions for which a given equation admits a unique solution in a certain ball. The main obtained result refers to the convergence of the modified chord method.

1. INTRODUCTION

Let X be a Banach space and $D \subseteq X$. Given $x_0 \in D$, we consider the ball $S(x_0,\rho) = \{x \in X \mid \|x - x_0\| \leq \rho\}$, with $\rho \in \mathbb{R}$, $\rho > 0$ and we assume that $S(x_0,\rho) \subseteq D$.

In this paper we shall study certain aspects of the problems regarding the existence, uniqueness, and the approximation of the solution of an operatorial equation of the form

$$f(x) = 0 \tag{1.1}$$

where $f : D \to X$.

Let $A : X \to X$ be a linear and continuous operator. For the approximation of the solution $\overline{x}$ of equation (1.1) we shall consider the sequence $(x_n)_{n\geq 0}$ given by

$$x_{n+1} = x_n - Af(x_n),\ n = 0, 1, ...,\ x_0 \in D. \tag{1.2}$$

In the following we shall obtain conditions under which equation (1.1) has a unique solution in the ball $S(x_0,\rho)$, and, moreover, the sequence $(x_n)_{n\geq 0}$ generated by (1.2) is convergent and $\lim x_n = \overline{x}$.

We shall apply the obtained results to the unified study of the convergence of some sequences of the form (1.2), namely generated by the modified Newton method, and the modified chord method. We obtain a result for the modified chord method and a known result for the modified Newton method.

2. EXISTENCE, UNIQUENESS AND APPROXIMATION

Regarding the existence and uniqueness of solution of equation (1.1) in the ball $S(x_0,\rho)$, and the convergence of sequence (1.2), the following result holds:

Theorem 2.1. *If $x_0 \in D$, ρ, A and f verify the following assumptions:*

i$_1$. $S(x_0,\rho) \subseteq D$;
ii$_1$. *application A is invertible;*
iii$_1$. *there exists $q \in \mathbb{R}$, $0 < q < 1$ such that $\forall x, y \in S(x_0,\rho)$ one has:*

$$\left\|A\left[f(x) - f(y) - A^{-1}(x-y)\right]\right\| \leq q\,\|x-y\|\,; \tag{2.3}$$

iv$_1$. *the following relation holds:*

$$\|Af(x_0)\| \leq \rho(1-q), \tag{2.4}$$

then the following hold true:

j$_1$. *equation (1.1) has a unique solution $\overline{x} \in S(x_0,\rho)$;*
jj$_1$. *sequence $(x_n)_{n\geq 0}$ generated by (1.2) is convergent and $\lim x_n = \overline{x}$;*
jjj$_1$. *the following hold true:*

$$\|\overline{x} - x_n\| \leq \frac{q^n}{1-q}\,\|x_1 - x_0\|\,,\ n = 1, 2, ...,; \tag{2.5}$$

$$\|\overline{x} - x_{n+1}\| \leq \frac{q}{1-q}\,\|x_{n+1} - x_n\|\,,\ n = 0, 1, ..., \tag{2.6}$$

Proof. Consider the operator $h : X \to X$ given by

$$h(x) = x - Af(x). \tag{2.7}$$

In the following we shall prove that the image of the ball $S(x_0,\rho)$ by h remains in $S(x_0,\rho)$, and moreover, h is a contraction on this ball.

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In this sense we notice that h may be written as

$$h(x) = x_0 - Af(x_0) - A\left[f(x) - f(x_0) - A^{-1}(x - x_0)\right]. \tag{2.8}$$

By (2.8) and (2.3) for $y = x_0$ and by $\mathbf{iv}_1$. we get

$$\|h(x) - x_0\| \le \rho(1-q) + q\|x - x_0\| \le \rho, \tag{2.9}$$

for all $x \in S(x_0, \rho)$. By (2.9) it follows that for all $x \in S(x_0, \rho)$, $h(x)$ given by (2.7) belongs to $S(x_0, \rho)$.

Consider now $y_1, y_2 \in S$. By (2.7) we get:

$$\begin{aligned} h(y_2) - h(y_1) &= y_2 - y_1 + A\left[f(y_1) - f(y_2)\right]) \\ &= A\left[f(y_1) - f(y_2) - A^{-1}(y_1 - y_2)\right]. \end{aligned}$$

whence, taking into account (2.3) it follows

$$\|h(y_2) - h(y_1)\| \le q\|y_2 - y_1\| \text{ for all } y_1, y_2 \in S(x_0, \rho),$$

i.e., h is a contraction on $S(x_0, \rho)$. Consequently, h has a unique fixed point $\overline{x} \in S(x_0, \rho)$. Since A is a linear invertible operator by (2.3) it follows that equation (1.1) has the unique solution $\overline{x} \in S(x_0, \rho)$.

Properties $\mathbf{jj}_1$ and $\mathbf{jjj}_1$ are immediate consequences of the fact that X is a Banach space and h is a contraction on the invariant ball $S(x_0, \rho)$ (see [1], [4]). Denoting $\rho = \|Af(x_0)\|$ then it can easily seen that

$$\frac{p}{1+q} \le \|\overline{x} - x_0\| \le \frac{p}{1-q}. \tag{2.10}$$

These relations are obtained from (2.8) for $x = \overline{x}$ and $\overline{x} = h(\overline{x})$.

From (2.10), if we replace $x_0 = x_n$ and $p = \|Af(x_n)\|$ we obtain lower and upper error bounds at the n-th step:

$$\frac{\|Af(x_n)\|}{1+q} \le \|\overline{x} - x_n\| \le \frac{\|Af(x_n)\|}{1-q} \tag{2.11}$$

The last relations offer a posteriori error bounds. □

3. The modified chord method

We consider now the divided differences of f at some given points in D. Let $\mathcal{L}(X)$ denote the set of linear operators from X to X.

Definition 3.1. [3] The linear operator $\alpha(x, y) \in L(X)$ is called first order divided difference of f at the points $x, y \in X$ if

a) $\alpha(x, y)(x - y) = f(x) - f(y)$;
b) if f is Fréchet differentiable at y then $\alpha(y, y) = f'(y)$.

We shall use in the following the usual notation $\alpha(x, y) = [y, x; f]$. In order to approximate the solution of equation (1.1) we consider the sequence $(x_n)_{n \ge 0}$ given by

$$x_{n+1} = x_n - [x_0, \overline{x}_0; f]^{-1} f(x_n), \; n = 0, 1, ..., \; x_0, \overline{x}_0 \in D. \tag{3.12}$$

We obtain the following result.

Theorem 3.2. *If $x_0, \overline{x}_0 \in D$, ρ and f verify the conditions*

i_2. $\overline{x}_0 \in S(x_0, \rho)$;
ii_2. *the linear operator $[x_0, \overline{x}_0; f]$ is invertible*
iii_2. *there exists $q \in \mathbb{R}$, $0 < q < 1$ such that for all $x, y \in S(x_0, \rho)$ one has*

$$\left\|[x_0, \overline{x}_0; f]^{-1}\left\{[y, x; f] - [x_0, \overline{x}_0; f]\right\}\right\| \le q; \tag{3.13}$$

the following relation holds:

$$p_1 = \left\|[x_0, \overline{x}_0 : f]^{-1} f(x_0)\right\| \le \rho(1-q),$$

then the following hold true:

j_2. *equation (1.1) has a unique solution $\overline{x} \in S$;*
jj_2. *sequence $(x_n)_{n\ge 0}$ generated by (3.12) is convergent and* $\lim x_n = \overline{x}$;
jjj_2. *the following relations are true:*

$$\|\overline{x} - x_n\| \le \frac{q^n}{1-q}\|x_1 - x_0\|, \; n = 1, 2, ..., ; \tag{3.14}$$

$$\|\overline{x} - x_{n+1}\| \le \frac{q}{1-q}\|x_{n+1} - x_n\|; \; n = 0, 1, ..., \tag{3.15}$$

$$\frac{p_1}{1+q} \le \|\overline{x} - x_0\| \le \frac{p_1}{1-q}; \tag{3.16}$$

$$\left\| \frac{[x_0, \overline{x}_0; f]^{-1} f(x_n)}{1+q} \right\| \le \|\overline{x} - x_n\| \le \le \frac{\left\|[x_0, \overline{x}; f]^{-1} f(x_n)\right\|}{1-q} < n = 0, 1, ..., \tag{3.17}$$

Proof. We shall prove that this result is a consequence of Theorem 2.1 and of relations (2.10) and (2.11). In this sense it suffices to show that relation (2.3) follows from (3.13) for $A = [x_0, \overline{x}_0; f]^{-1}$.

The definition of the divided differences, leads to the equality

$$\begin{aligned}&[x_0, \overline{x}_0; f]^{-1} \{f(x) - f(y) - [x_0, \overline{x}_0; f](x-y)\}\\ &= [x_0, \overline{x}_0; f]^{-1} \{[y, x; f] - [x_0, \overline{x}_0; f]\} (x-y),\end{aligned}$$

whence, by (3.13), it follows for $A = [x_0, \overline{x}_0; f]^{-1}$,

$$\begin{aligned}&\left\|[x_0, \overline{x}_0; f]^{-1} [f(x) - f(y) - [x_0, \overline{x}_0; f](x-y)]\right\|\\ &\le q\, \|x - y\| .\end{aligned}$$

It is obvious that Theorem 3.2 is a consequence of Theorem 2.1 and of relations (2.10) and (2.11). □

4. The modified Newton method

Assume that f is Fréchet differentiable at all points $x \in D$. Let $x_0 \in D$ and $f'(x_0) \in \mathcal{L}(X)$. Assume that $f'(x_0)$ is invertible.

In order to approximate $\overline{x}$ we shall consider the sequence $(x_n)_{n \ge 0}$ generated by the modified Newton method.

$$x_{n+1} = x_n - [f'(x_0)]^{-1} f(x_n),\ x_0 \in D,\ n = 0, 1, ... \tag{4.18}$$

Concerning the convergence of this sequence the following result holds:

Theorem 4.3. [2] *If $x_0 \in D$, $\rho \in \mathbb{R}$, $\rho > 0$ and f verify*

- i$_3$. $S(x_0, \rho) \subseteq D$;
- ii$_3$. *f is Fréchet differentiable at all points $x \in S(x_0, \rho)$, and $f'(x_0)$ is a linear continuous operator;*
- iii$_3$. *the operator $f'(x_0)$ is invertible*
- iv$_3$. *there exists $q \in \mathbb{R}$, $0 < q < 1$ such that for all $x \in S(x_0, \rho)$ one has.*

$$\left\|[f'(x_0)]^{-1} [f'(x) - f'(x_0)]\right\| \le q; \tag{4.19}$$

- v$_3$. *one has*

$$\overline{p} = \left\|[f'(x_0)]^{-1} f(x_0)\right\| \le \rho(1-q),$$

then the following relations hold:

- j$_3$. *equation (1.1) has a unique solution $\overline{x} \in S(x_0, \rho)$;*
- jj$_3$. *the sequence $(x_n)_{n \ge 0}$ given by (4.18) is convergent and $\lim x_n = \overline{x}$;*
- jjj$_3$. *the following estimations hold true:*

$$\|\overline{x} - x_n\| \le \frac{q^n}{1-q} \|x_1 - x_0\|,\ n = 1, 2, ..., \tag{4.20}$$

$$\|\overline{x} - x_{n+1}\| \le \frac{q}{1-q} \|x_{n+1} - x_n\|,\ n = 0, 1, ..., \tag{4.21}$$

$$\frac{\overline{p}}{1+q} \le \|\overline{x} - x_0\| \le \frac{\overline{p}}{1-q}. \tag{4.22}$$

$$\frac{\left\|[f'(x_0)]^{-1} f(x_n)\right\|}{1+q} \le \|\overline{x} - x_n\| \le \frac{\left\|f'(x_0)]^{-1} f(x_n)\right\|}{1-q}. \tag{4.23}$$

Proof. It is sufficient to show that relation (4.19) implies relation (2.3) for $A = [f'(x_0)]^{-1}$.

Let $g : S(x_0, \rho) \to X$ be an operator differentiable at all points $x \in S(x_0, \rho)$. Then it is known (see, e.g.,[2]) that for all $x, y \in S(x_0, \rho)$ one has

$$\|g(x) - g(y)\| \le \|x - y\| \sup_{0<\theta<1} \|f'(y + \theta(x-y))\| . \tag{4.24}$$

Let now take g given by

$$g(x) = [f'(x_0)]^{-1} [f(x) - f'(x_0)x], \tag{4.25}$$

then

$$g'(x) = [f'(x_0)]^{-1} [f'(x) - f'(x_0)]. \tag{4.26}$$

If we take into account (4.25) and (4.26), by (4.24) we get

$$\begin{aligned}&\left\|[f'(x_0)]^{-1}[f(x)-f(y)-f'(x_0)(x-y)]\right\|\\&\leq \|x-y\| \sup . \left\|[f'(x_0)]^{-1}[f'(y+\theta(x-y))-f'(x_0)]\right\|\\&\leq q\,\|x-y\|\,,\end{aligned}$$

which shows that (2.3) is verified for $A = f'(x_0)^{-1.}$ Theorem 4.3 is a consequence of Theorem 2.1, and therefore properties $\mathbf{j}_3$–$\mathbf{jjj}_3$ are true. □

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